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GENERALIZATION OF SOME INVESTIGATIONS OF A. M. LIAPUNOV ON LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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1. In the present article is studied a system of n=2m differential equations, which in vector-matrix notation may be written in the following form: /445*

$$\frac{dx}{dt} = \lambda JH(t)x, \tag{1}$$

where x=x(t) is an n-dimensional vector function; λ is some parameter; $H(t)=\|h_{jk}(t)\|_1^n$ is a real symmetric matrix, the elements of which are the sums of periodic functions of "time" t: $H(t+\omega)=H(t)(-\infty<t<\infty)$, and

$$J=\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \quad I_m=\begin{Vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{Vmatrix}.$$

Any system of canonical equations of a mechanical system with m degrees of freedom is of the form (1) in the case when the Hamiltonian of this system is a quadratic form of generalized coordinates $q_1, q_2, \ldots, q_m$ and generalized impulses $p_1, p_2, \ldots, p_m$, with periodic functions of t for coefficients, for all that $x=(q_1, \ldots, q_m, p_1, \ldots, p_m)$, while λ may be considered to be equal to one.

[†]Presented by Academician A. N. Kolmogorov on May 22, 1950.

*Numbers in right margin indicate pagination of foreign text.

Any system of second order differential equations of the form:

$$\frac{d^2y}{dt^2} + \lambda^2 P(t) y = 0, \tag{2}$$

where $y=y(t)$ is an m-dimensional vector-function, and $P(t)=P(t+\omega)$ is a symmetric matrix-function, may be reduced to system (1).

Namely, if one denotes x to be the direct sum of the vectors y and $\lambda^{-1}dy/dt$, then the vector $x=(y, \lambda^{-1}dy/dt)$, by virtue of (2), will satisfy equation (1), in which the matrix H(t) is the direct sum of matrices P(t) and I_m.

Let $U=\|u_{jk}(t; \lambda)\|_1^n$ be a matrix-function, which satisfies the following system:

$$\frac{dU}{dt} = \lambda JHU, \quad U(0;\lambda) = I_n.$$

By a well-known theorem of Liapunov-Poincare (see [1], p.226)/446 the characteristic polynomial $\det(U/\omega; \lambda)-\rho I_n)$ of the matrix $U(\omega;\lambda)$ is recursive. Its roots $p_1(\lambda)$, ..., $p_n(\lambda)$ are called multiplicators.

All the solutions of equation (1) for a given λ will be bounded on the interval $(0, \infty)$ in the one and only one case if, given this λ, the absolute value of all the multiplicators is equal to 1 and all the elementary factors of the matrix are linear.

The open interval on the real axis, $(\alpha,\beta)(-\infty\leq\alpha, \beta\leq\infty)$, is called the zone of stability, if for all values of λ on this interval all the solutions of equation (1) are bounded and no larger interval, containing (α,β), possesses this property.

The problem of determining the zone of stability of equation (1) plays a vital role in various questions of mechanics and radio engineering (in questions of parametric resonance, dynamic stability and others).

For the case of scalar equation (2), the main results of the existence of zones of stability, their arrangements, as well as the methods for their determination were obtained by A. M. Liapunov [1-6]. Despite the importance of the task of generalizing all of these remarkable investigations of A. M. Liapunov for the case of vector equations, we have no knowledge of anything essential concerning this question, besides that which was pointed out in [1] by A. M. Liapunov himself.

Here are brought in some of the results of our investigations in this direction.

2. Further on, not stipulating it, we will assume satisfaction of the condition:

A. For any t, the form

$$(H(t)\xi,\xi)=\sum_{1}^{n}h_{jk}(t)\xi_k\xi_j \quad (\xi\neq 0) \tag{3}$$

is non-negative, and its mean on argument t is positive in the interval $(0,\omega)$.

With this assumption the theorem of Liapunov-Poincare can be supplemented with the following proposition.

Theorem 1. For any imaginary λ, m multiplicators of equation (1) lie within the unit circle, and the other m lie without.

We will examine the importance of λ in the upper half-plane: $I_m\lambda\leq 0$. The multiplicator $\rho(\lambda)$ we will call a multiplicator of the first or second class depending upon whether it lies within the unit circle or it lies without. Let a be some point on the upper half-plane, in whose neighborhood the multiplicators of the first class $\rho_1(\lambda)$, $\rho_2(\lambda)$, ..., $\rho_m(\lambda)$ can be determined as single-valued analytic functions. Let one of these multiplicators $\rho(\lambda)$, having been analytically extended along a path, from a point a and some real point α which is not a branch point, fall on the unit circle $|\rho(\alpha)|=1$. Since under the mapping $\lambda\to\rho(\lambda)$, the upper semicircle of points α is mapped onto the lower part of the unit circle, then $\rho'(\alpha)\neq 0$, and if the complex number $\rho'(\alpha)$ is represented by a vector with the origin at point $\rho(\alpha)$, then this vector will touch the unit circle and will be directed counterclockwise ($i\rho'(\alpha)/\rho(\alpha)<0$). We come to the following simple but important theorem.

Theorem 2. If by the increasing of parameter λ from real α to some $\beta>\alpha$, some multiplicator of the first class moves along the unit circle, then its movement occurs monotonically in a countercloskwise direction.

Let us examine a boundary value problem: /447

$$\frac{dx}{dt} = \lambda JH(t)\,x, \quad x(0) = -x(\omega). \tag{4}$$

By virtue of theorem 1, the characteristic numbers of this problem are real. It can be shown that among them are always some positive reals; let Λ_0 be the minimum of these.

It is not difficult to show that

$$\Lambda_0 > 2\left(\int_0^{\omega}\eta(t)\,dt\right)^{-1},$$

where $\eta(t)$ is the maximum of the characteristic numbers of the matrix $H(t)$.

After a series of sufficiently subtle arguments concerning that which occurs when multiplicators of the same class and of different classes meet in their movements along the unit circle, it is necessary to establish a theorem:

Theorem 3. The interval $0<\lambda<\lambda_0$ belongs to the zone of stability of equation (1).

4. If equation (1) is obtained through a transformation of equation (2) as pointed out in part 1, then the number Λ_0^2 will be

the minimum characteristic number of the boundary value problem

$$\frac{d^2y}{dt^2} + \mu P(t)\, y = 0; \quad y(0) = -y(\omega), \quad y'(0) = -y'(\omega). \tag{5}$$

Let $y_0(t) \neq 0$ be some solution of this system, given $\mu = \Lambda_0^2$. It apparently possesses the property $y_0(t+\omega) = -y_0(t)$ $(-\infty < t < \infty)$. Let us set

$$R = \max |y_0(t)|, \quad V = \max |y_0'(t)|.$$

Let us explain that if $u = (u_1, \ldots, u_n)$ is a vector, then $|u|$ designates its Eucledian length.

The length L of the arc $y = y_0(t)$ $(t \leq t \leq t+\omega)$ in m-dimensional space apparently does not depend on the choice of the number τ. On the other hand, it is not less than the distance between its origin $y_0(t)$ and its end point $y_0(t+\omega) = -y_0(t)$.

From here

$$2R \leqslant L = \int_0^\omega \left| \frac{dy_0}{dt} \right| dt \leqslant V\omega. \tag{6}$$

Let τ now be chosen in such a way that $V = |y'(\tau)|$. Integrating both sides of equation (5) within the limits τ and $\tau+\omega$, we will find

$$2V = 2|y'(\tau)| = \Lambda_0^2 \left| \int_\tau^{\tau+\omega} P(t)\, y(t)\, dt \right| \leqslant \Lambda_0^2 R \int_\tau^{\tau+\omega} \pi(t)\, dt, \tag{7}$$

where $\pi(t)$ is the maximum characteristic number of the matrix P(t).

Comparing (6) and (7), we obtain a minimum estimate for Λ_0^2, from which:

<u>Theorem 4</u>. If the symmetric matrix P(t) satisfies condition A, then for

$$0 < \lambda^2 < \frac{4}{\omega} \left(\int_0^\omega \pi(t)\, dt \right.$$

the solutions to equation (2) are bounded.

For the case m=1, theorem 4 coincides precisely with the well/448 known theorem of A. M. Liapunov (see [1], p.217).

One can prove theorem 4 under more general conditions relative to the matrix P(t), such that it embraces the more recent generalization of the theorems of A. M. Liapunov, obtained by Borg [7] for the scalar equation (2).

5. For m=1 (n=2) system (1) will have two multiplicators. One $\rho(\lambda)$ from the first class and the second $\rho^{-1}(\lambda)$ from the second class ($\mathrm{Im}\lambda \geq 0$). The investigation of their movements under λ, changing from 0 to ∞, brings one to such inequalities:

$$0<\Lambda_0\leqslant\Lambda_1<\lambda_1\leqslant\lambda_2<\Lambda_2\leqslant\Lambda_3<\lambda_3\leqslant\lambda_4<\ldots \tag{8}$$

Here $\Lambda_0 \leq \Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \Lambda_4 < \ldots$ are positive characteristic numbers of the boundary value problem (4), while $\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4$ ... are positive characteristic numbers of the boundary problem:

$$\frac{dx}{dt}=\lambda JHx, \quad x(0)=x(\omega). \tag{9}$$

For the case of the scalar equation (2), the inequalities (8) were first established by A. M. Liapunov [4], who came to them via another route.

For n=2, a more complete theorem than theorem 2 can be formulated.

<u>Theorem 5</u>. For a continuous increase of λ from Λ_{2k+1} to λ_{2k-1} (from λ_{2k} to Λ_{2k}), the multiplicator $\rho(\lambda)$ moves counterclockwise along the half circle from -1 to 1 (from 1 to -1).

For the case of the scalar equation (2), the theorem was established by Putnam via another route.

A theorem analogous to theorem 5 may also be expressed with respect to negative characteristic numbers of boundary value problems (4) and (8).

For the case $n=2m>2$, we succeeded in showing the existence of an infinite number of zones of stability of equation (1), moving away in both directions towards infinity, only under the assumption that the matrix $H(t)$ is twice continuously differentiable and that all of its characteristic numbers are positive and distinct for any t.

Under more general assumptions, the question remains unanswered.

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16. Abstract

The article is a study of a system of n = 2m differential equations of the form

$$\frac{dx}{dt} = \lambda J H(t) x.$$

In particular, it describes the results of recent research done on the solutions of these differential equations, based upon the work of A.M. Liapunov.

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